§ 6 Applications of Differentiation

6.1 Rolle's Theorem and Mean Value Theorem

Theorem 6.1.1

Let $f:(a,b)\to\mathbb{R}$ be a function and $c\in(a,b)$ such that

1) $f'(c)$ exists

2) f attains maximum (or minimum) at $x=c$.

Then, we have $f'(c)=0$.

proof: Assume f attains maximum at $x=c$

$$f'(c) \text{ exist } \Rightarrow \lim_{\Delta x\to 0^+}\frac{f(c+\Delta x)-f(c)}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{f(c+\Delta x)-f(c)}{\Delta x}=f'(c)$$

Note: $\dfrac{f(c+\Delta x)-f(c)}{\Delta x}\leq 0$ for all $\Delta x>0 \Rightarrow f'(c)=\lim\limits_{\Delta x\to 0^+}\dfrac{f(c+\Delta x)-f(c)}{\Delta x}\leq 0$

$\dfrac{f(c+\Delta x)-f(c)}{\Delta x}\geq 0$ for all $\Delta x<0 \Rightarrow f'(c)=\lim\limits_{\Delta x\to 0^+}\dfrac{f(c+\Delta x)-f(c)}{\Delta x}\geq 0$

$\therefore f'(c)=0.$

Theorem 6.1.2 (Rolle's Theorem)

Let $F:[a,b]\to\mathbb{R}$ be a function such that

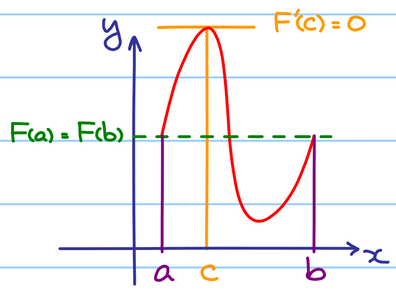
1) F is continuous on $[a,b]$

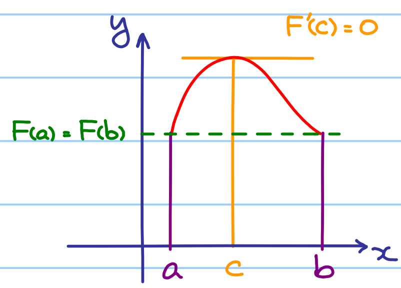
2) F is differentiable on (a,b)

3) $F(a)=F(b)$

then there exists $c\in(a,b)$ such that $F'(c)=0$.

Geometrical meaning:





Idea of proof:

By the Maximum-Minimum Theorem, there exist $x_m, x_M \in [a,b]$ such that $F(x_m) \le F(x) \le F(x_M)$ for all $x \in [a,b]$.

Case 1: Either x_m or x_M lies on (a,b)

then $F'(x_m) = 0$ or $F'(x_M) = 0$ (Need some argument)

Case 2: Both x_m and x_M lies on boundary points of $[a,b]$,

i.e. $x_m = a$, $x_M = b$ or $x_m = b$, $x_M = a$

By assumption, $F(a) = F(b)$ which forces that $F(x)$ is constant on $[a,b]$

so $f'(x) = 0$ for all $x \in (a,b)$

Theorem 6.1.3 (Mean Value Theorem)

Let $f: [a,b] \to \mathbb{R}$ be a function such that

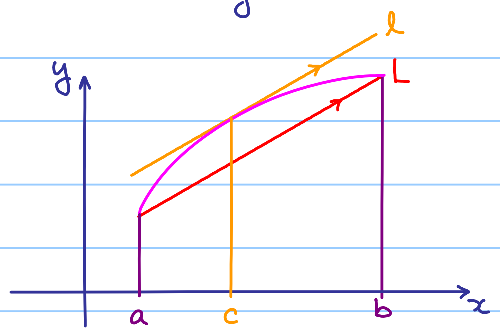
1) f is continuous on $[a,b]$

2) f is differentiable on (a,b)

then there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

slope of ℓ slope of L.

Geometrical meaning:



Idea of proof:

Looking for $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

i.e. looking for a solution in (a,b) of the equation

$$\underbrace{f'(x) - \frac{f(b)-f(a)}{b-a}} = 0.$$

Idea: Realize this as $F(x)$ and apply Rolle's theorem.

proof :

Let $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

Check : 1) F is continuous on $[a,b]$

2) F is differentiable on (a,b)

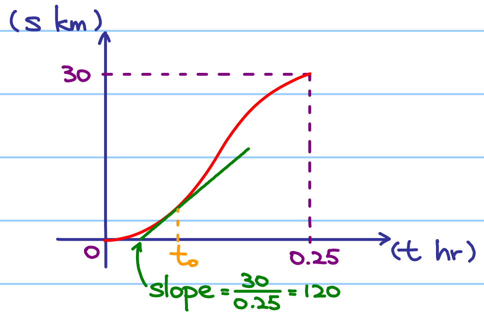
3) $F(a) = F(b) = 0$

Apply Rolle's Theorem to F, the result follows.

Question :

A vehicle is speeding on a highway if its speed ≥ 120 km/hr (at some moment)

If the length of the highway is 30 km and if kelvin only spent 15 minutes on the highway. Should he be arrested ?



By the MVT, there exists $t_0 \in (0, 0.25)$ such that slope of the tangent at $t = t_0$ $= \frac{30}{0.25} = 120$

i.e. instantaneous speed at $t = t_0$ $= 120$ km/hr

6.2 Applications of Mean Value Theorem

Theorem 6.2.1

If $f: \mathbb{R} \to \mathbb{R}$ is a differentiable and $f'(x) = 0 \ \forall x \in \mathbb{R}$,

then $f(x)$ is a constant function.

proof : Fix $x_0 \in \mathbb{R}$, let $x \in \mathbb{R} \setminus \{x_0\}$

If $x > x_0$, note f is differentiable everywhere (in particular, on (x_0, x))

$\Rightarrow f$ is continuous everywhere (in particular, on $[x_0, x]$)

Apply MVT, $\exists\, c \in (x_0, x)$ such that

$$f(x_0) - f(x) = \underbrace{f'(c)}_{\substack{\| \\ 0 \text{ by assumption.}}}(x - x_0) = 0$$

i.e. $f(x) = f(x_0) \quad \forall x > x_0$

We have similar result if $x < x_0$, the result follows.

Example 6.2.1

Let $f(x) = \cos^2 x + \sin^2 x$

$f'(x) = -2\cos x \sin x + 2\sin x \cos x = 0$

$\therefore \cos^2 x + \sin^2 x$ is a constant.

In particular, $f(0) = 1$, so $f(x) = \cos^2 x + \sin^2 x = 1$

Theorem 6.2.2

If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$, then $f(x) = g(x) + C$, where C is a constant.

proof: Let $h(x) = f(x) - g(x)$.

Then $h'(x) = f'(x) - g'(x) = 0$

$\therefore h(x) = C$, where C is a constant. i.e. $f(x) = g(x) + C$.

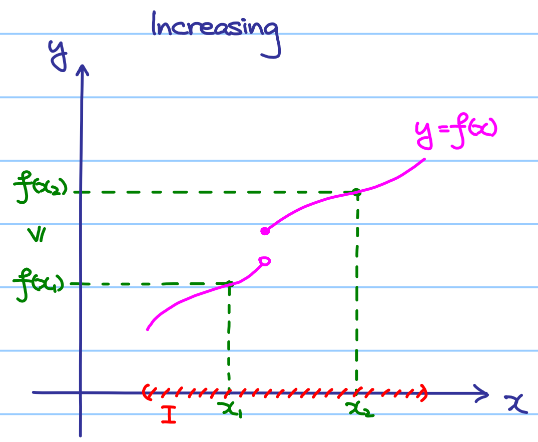
Next, we are going to discuss how differentiation helps to find maximum / minimum points of a function.

Firstly, we make some preparations:

6.3 Increasing / Decreasing Functions

Definition 6.3.1

Let I be an interval and let $f : I \to \mathbb{R}$ be a function such that $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$) then $f(x)$ is called an increasing (a decreasing) function.†



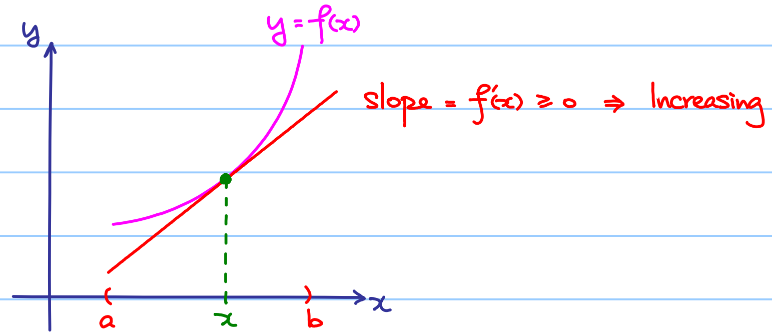
Roughly speaking:
The larger x we input the larger y we get!

† If we have a strictly inequality, it is called a strictly increasing (decreasing) function.

Theorem 6.3.1

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function.

If $f'(x)\geq 0$ ($f'(x)\leq 0$) for all $x\in(a,b)$ then f is an increasing (decreasing) function.[++]



++ If we have strict inequality, $f(x)$ is a strictly increasing (decreasing) function on (a,b).

proof:

If $a<x_1<x_2<b$,

apply MVT to f on $[x_1,x_2]$,

$\exists\, c\in(x_1,x_2)$ such that $f(x_2)-f(x_1) = \underbrace{f'(c)}_{\geq 0}\,\underbrace{(x_2-x_1)}_{>0} \geq 0$

By assumption

Example 6.3.1

$f(x) = -5x^2+80x-120$

$f'(x) = -10x+80$

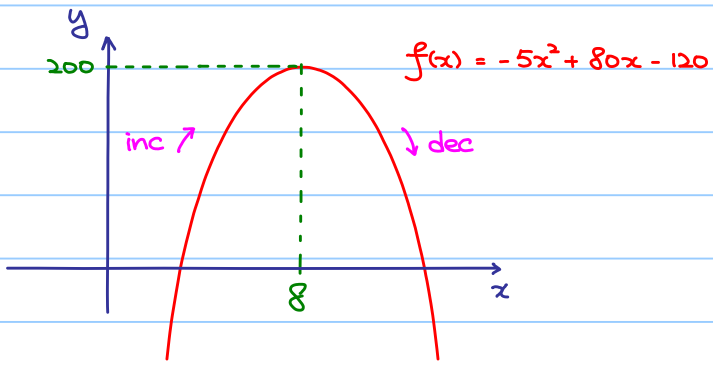
$f'(x)>0$	$f'(x)<0$
$-10x+80>0$	$-10x+80<0$
$x<8$	$x>8$

$\therefore$ $f(x)$ is strictly increasing when $x<8$ and

$f(x)$ is strictly decreasing when $x>8$.

Not hard to understand why $f(x)$ attains maximum when $x=8$

and maximum value $= f(8) = 200$



Note : $f'(8) = 0$

Remark : Verify the answer by using completing square.

Question :

1) If $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$,
   is it enough to say that f attains maximum at $x = a$?
2) If we want to find all extrema,
   is it enough to solve $f'(x) = 0$?

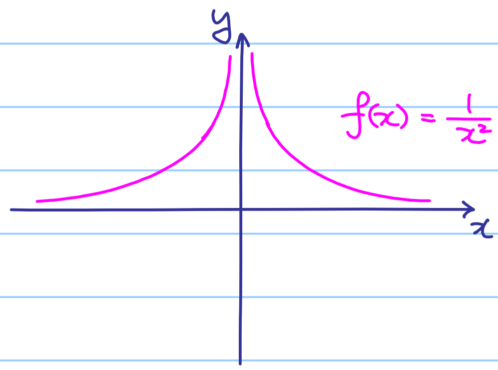
Example 6.3.2

Let $f(x) = \frac{1}{x^2}$, $x \neq 0$.

$f'(x) = -\frac{2}{x^3}$

$f'(x) > 0$ for $x < 0$

$f'(x) < 0$ for $x > 0$



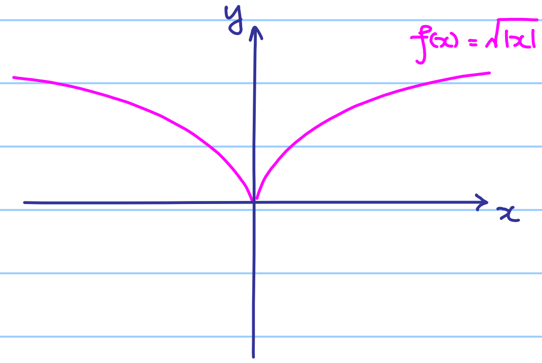
$\therefore$ $f(x)$ is strictly increasing when $x < 0$

$f(x)$ is strictly decreasing when $x > 0$

However, $f(0)$ is NOT well-defined, so there is NO maximum point.

Example 6.3.3

Let $f(x)=\sqrt{|x|}$



Rewrite :

$$f(x)=\begin{cases}\sqrt{x} & \text{if } x>0\\ 0 & \text{if } x=0\\ \sqrt{-x} & \text{if } x<0\end{cases}$$

If $x>0$, $f(x)=\sqrt{x}$, then $f'(x)=\frac{1}{2\sqrt{x}}>0$

If $x<0$, $f(x)=\sqrt{-x}$, then $f'(x)=-\frac{1}{2\sqrt{-x}}<0$

$\therefore$ $f(x)$ is strictly increasing when $x>0$

$f(x)$ is strictly decreasing when $x<0$

However, $\lim\limits_{\Delta x\to 0^+}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim\limits_{\Delta x\to 0^+}\frac{\sqrt{\Delta x}}{\Delta x}=\lim\limits_{\Delta x\to 0^+}\frac{1}{\sqrt{\Delta x}}$ which does NOT exist ,

$\Rightarrow$ $\lim\limits_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}$ does NOT exist

$\Rightarrow$ $f'(0)$ does NOT exist

but as we can see f still attains minimum at $x=0$.

$\therefore$ Solving $f'(x)=0$ to find max / min is NOT enough.

Answers for both questions 1 and 2 are negative ,

so, what is the exact statement of finding an extrema ?

6.4 First Derivative Check

Theorem 6.4.1

Let I be an open interval and let $a\in I$.

Let $f: I\to\mathbb{R}$ be a function such that

1) f is continuous

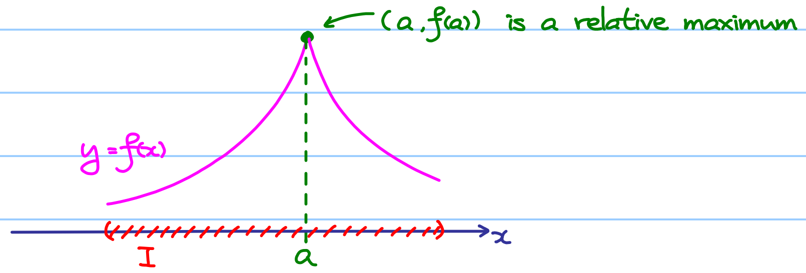
2) $f'(x)\geq 0$ ($f'(x)\leq 0$) for all $x\in I$ with $x<a$

3) $f'(x)\leq 0$ ($f'(x)\geq 0$) for all $x\in I$ with $x>a$

Then $(a, f(a))$ is a relative maximum (minimum) .

Note : We do NOT require the differentiability of f at $x=a$, but only the continuity of f at $x=a$.

Geometrical meaning:



Remember the slogan: Change of sign of $f'(x)$ at $x=a$

proof:

Let $x \in I$ and $x < a$.

Note: f is continuous on $[x,a]$ and

f is differentiable on (x,a)

apply the MVT, there exists $c \in (x,a)$ such that

$$f(a)-f(x) = \underbrace{f'(c)}_{\geq 0} \underbrace{(a-x)}_{> 0} \geq 0$$

By assumption

$\therefore f(x) \leq f(a)$ for all $x \in I$ with $x < a$

Similarly, we can also show that $f(x) \leq f(a)$ for all $x \in I$ with $x > a$

$\therefore f(x) \leq f(a)$ for all $x \in I$, i.e. $(a, f(a))$ is a relative maximum.

Example 6.4.1

Prove that $e^x \geq 1+x$ (i.e. $e^x - x - 1 \geq 0$) for all $x \in \mathbb{R}$.

Let $f(x) = e^x - x - 1$

(Want to find the global minimum of $f(x)$ and see if it is ≥ 0.)

$f'(x) = e^x - 1$

$f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$

f is strictly increasing when $x > 0$ and strictly decreasing when $x < 0$

(and f is continuous at $x=0$.)

f attains minimum when $x = 0$ (By 1st derivative check)

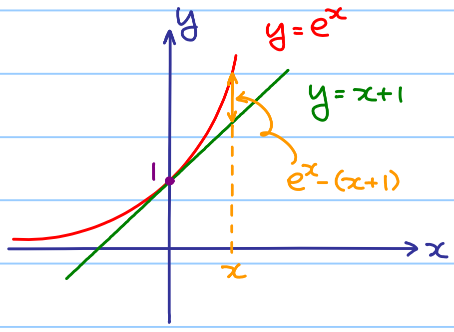
(In fact, global minimum, why?)

$\therefore\ f(x) \geq f(0) \quad \forall\, x \in \mathbb{R}$ ——— (*)

$= e^0 - 0 - 1$

$= 0$

Note: The equality holds iff $x = 0$



Definition 6.4.1

If $f'(a) = 0$, then $(a, f(a))$ is said to be a stationary point.

However, a stationary point is NOT necessary to be a relative extrema.

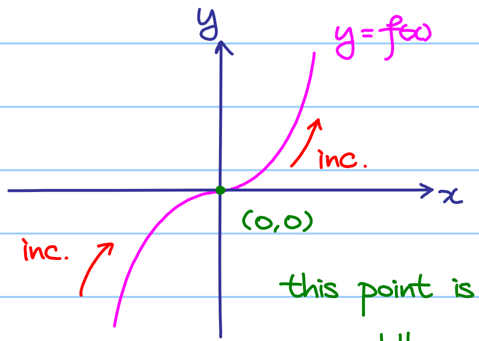
Example 6.4.2

If $f(x) = x^3$, then $f'(x) = 3x^2$

Note: 1) $f'(0) = 0$

2) $f'(x) = 3x^2 > 0$ for $x \neq 0$

i.e. No change of sign of $f'(x)$ at $x = 0$.



this point is called a saddle point

Note: a stationary point is NOT necessary to be a max./min. point!

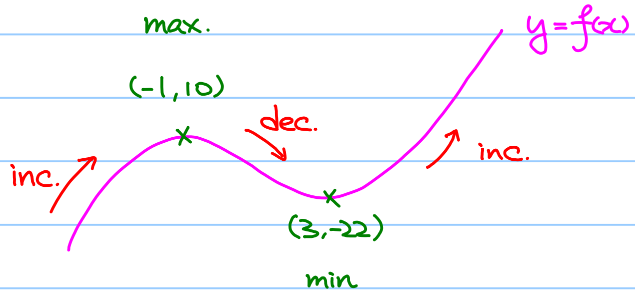
Example 6.4.3

If $f(x) = x^3 - 3x^2 - 9x + 5$

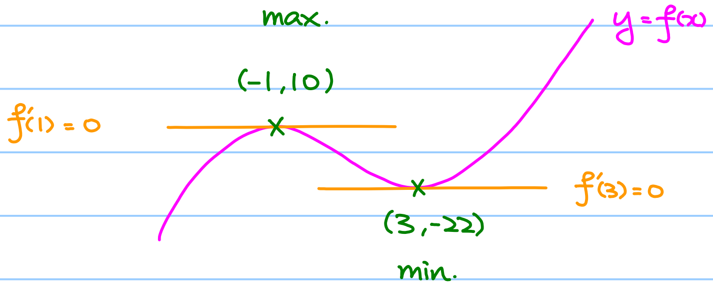
then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x-1)$

$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$



Furthermore,

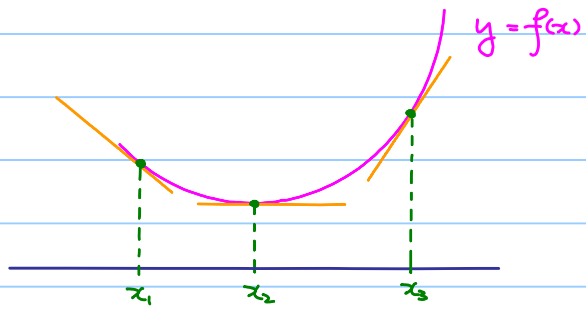


6.5 Second Derivative Check

Let I be an open interval.

$f''(x) > 0$ for $x \in I \Rightarrow f'(x)$ is strictly increasing.

Geometrical meaning:



Slope of the tangent line at $(x, f(x))$ increases as x increases!
(NOT $f(x)$ is increasing!)

Theorem 6.5.1

Let I be an open interval.

If $f''(x)>0$ ($f''(x)>0$) for all $x \in I$, then $f(x)$ is concave (convex) on I.

Theorem 6.5.2 (Second Derivative Check)

Let I be an open interval and let $a \in I$.

If $f: I \to \mathbb{R}$ be a function such that

1) $f'(a)=0$ (i.e. $(a, f(a))$ is a stationary point.)

2) $f''(a)<0$ ($f''(a)>0$) and $f''(x)$ is continuous at $x=a$ (i.e. f is convex (concave) near $x=a$)

then $(a, f(a))$ is a relative maximum (minimum).

Caution: If $f''(a) = 0$, then NO conclusion!

Consider $f(x) = x^4, x^3, -x^4$

We have $f'(0) = f''(0) = 0$ in each case, but $(0,0)$ is

- min. for the 1st case.
- saddle point for the 2nd case.
- max. for the 3rd case.

Example 6.5.1

If $f(x) = x^3 - 3x^2 - 9x + 5$

then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x-1)$

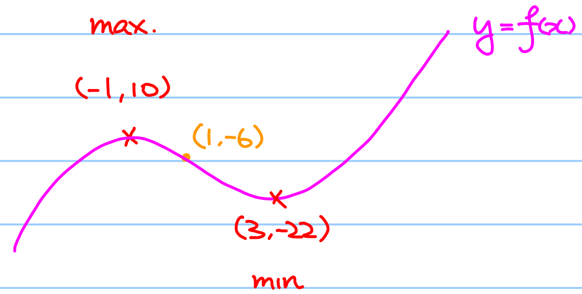
$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$

$f''(x) = 6x - 6$

$f''(x) > 0$ if $x > 1$ $\qquad f''(-1) = 12 < 0$

$f''(x) < 0$ if $x < 1$ $\qquad f''(3) = 12 > 0$



		-1		3	
$f'(x)$	+ve		-ve		+ve
$f(x)$	inc.		dec.		inc.

		1	
$f''(x)$	-ve		+ve
$f(x)$	convex		cancave

Note: The curve changes from being convex to concave at $(1,6)$.

This point is called a point of inflection.

Definition 6.5.1

Let I be an open interval and let $a \in I$.

Let $f: I \to \mathbb{R}$ be a function such that

1) f is continuous
2) $f''(x) > 0$ ($f''(x) < 0$) for all $x \in I$ with $x < a$
3) $f''(x) < 0$ ($f''(x) > 0$) for all $x \in I$ with $x > a$

then $(a, f(a))$ is said to be a point of inflection.

Example 6.5.2

$f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$

Find the range of x such that

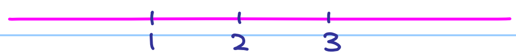
(1) $f'(x) > 0$, $f'(x) < 0$

(2) $f''(x) > 0$, $f''(x) < 0$

Step 1 : Find $f'(x)$ and factorize it.

$$f'(x) = 60x^4 - 420x^3 + 1020x^2 - 1020x + 360$$
$$= 60(x^4 - 7x^3 + 17x^2 - 17x + 6)$$
$$= 60(x-1)^2(x-2)(x-3)$$ (Using factor theorem)

Step 2:



gives intervals $x<1$, $1<x<2$, $2<x<3$, $x>3$

Reason : those factors may change sign at the boundary points of intervals.

Step 3:

	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$(x-1)^2$	+	0	+	+	+	+	+
$(x-2)$	−	−	−	0	+	+	+
$(x-3)$	−	−	−	−	−	0	+
$f'(x)$	+	0	+	0	−	0	+
$f(x)$	inc	saddle pt.	inc.	max.	dec.	min	inc.

saddle point $= (1, -23)$ max $= (2, -16)$ min $= (3, -39)$

Similarly,

$$f''(x) = 240x^3 - 1260x^2 + 2040x - 1020$$

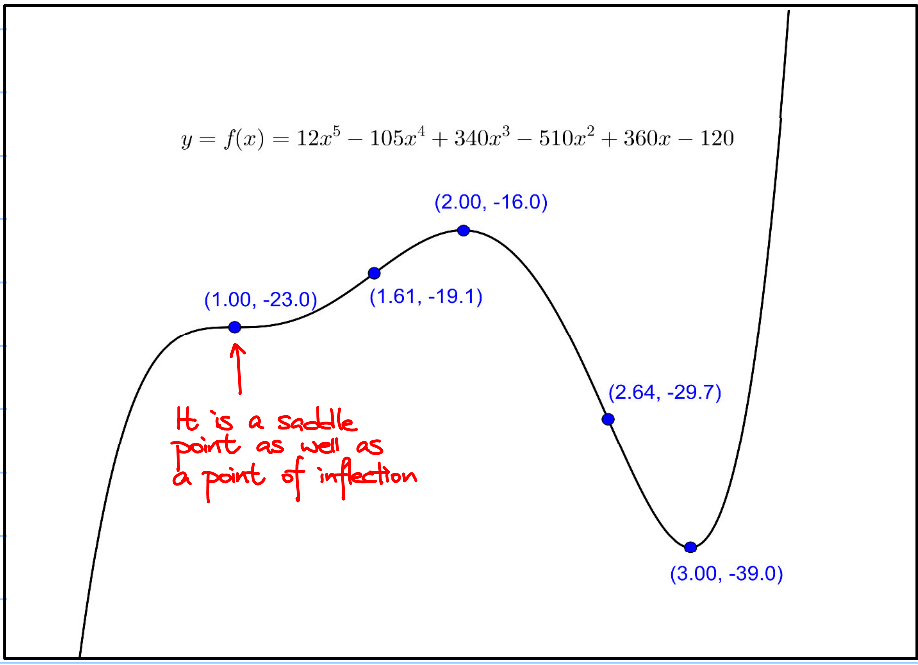
$$= 60(x-1)(4x^2-17x+17)$$

$$= 240(x-1)\left[x-\left(\frac{17+\sqrt{17}}{8}\right)\right]\left[x-\left(\frac{17-\sqrt{17}}{8}\right)\right]$$

$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
		1		$\frac{17-\sqrt{17}}{8} \approx 1.61$		$\frac{17+\sqrt{17}}{8} \approx 2.64$	
$f(x)$	$\cap$		$\cup$		$\cap$		$\cup$

points of inflection: $(1,-23)$, $\left(\frac{17\pm\sqrt{17}}{8}, f\left(\frac{17\pm\sqrt{17}}{8}\right)\right)$

$= (1.61, -19.1)$ or $(2.64, -29.7)$



Example 6.5.3

$f(x) = \dfrac{x}{(x+1)^2}$, $x \neq -1$.

$f'(x) = \dfrac{1-x}{(x+1)^3}$

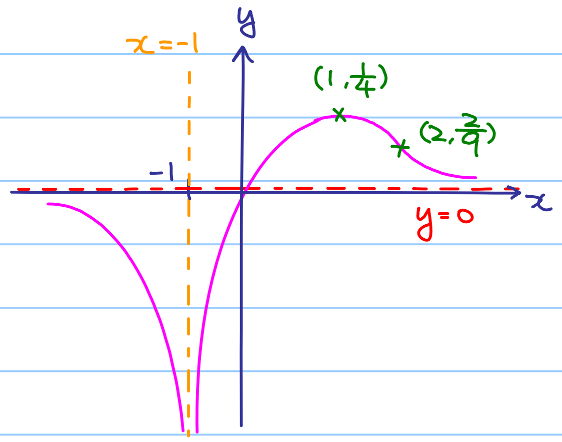
		-1		1	
$f'(x)$	$-$	NOT defined	$+$	0	$-$
$\Downarrow$					
$f(x)$	dec.	NOT defined	inc.	max.	dec.

max $= (1, \frac{1}{4})$

$f''(x) = \dfrac{2(x-2)}{(x+1)^4}$

		-1		2	
$f''(x)$	$-$	NOT defined	$-$		$+$
$\Downarrow$					
$f(x)$	$\cap$		$\cap$		$\cup$

point of inflection: $(2, \frac{2}{9})$



Note: The graph of $y = f(x)$ behaves like:

- the vertical line $x = -1$, when x is "near" -1.
- the horizontal line $y = 0$, when x is "near $+\infty$ or $-\infty$".

In fact, $x = -1$ is called a vertical asymptote,

$y = 0$ is called a horizontal asymptote.

6.6 Asymptotes

Definition 6.6.1

1) If $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x) = +\infty$ or $-\infty$,

then $x=a$ is said to be a vertical asymptote.

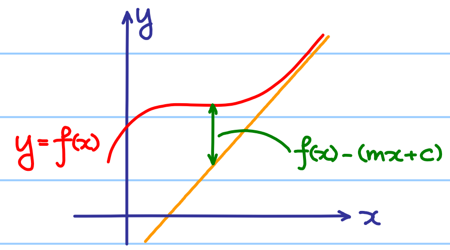
2) If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, where $L \in \mathbb{R}$,

then $y=L$ is said to be a horizontal asymptote.

Note : It may happen that both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist but they are NOT the same.

3) If $y=mx+c$ is a straight such that $\lim_{x\to+\infty} f(x)-(mx+c)=0$ or $\lim_{x\to-\infty} f(x)-(mx+c)=0$,

then the straight line is said to be an oblique asymptote of $f(x)$.



the distance tends to 0 as $x \to +\infty$

Example 6.6.1

Let $f(x) = \dfrac{x|x-2|}{x-1}$, $x \neq 1$.

$$f(x) = \begin{cases} \dfrac{x(x-2)}{x-1} & \text{if } x \geq 2 \\ -\dfrac{x(x-2)}{x-1} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

(a) Show that f is NOT differentiable at $x=2$.

Hint : Show that $\lim_{\Delta x\to 0} \dfrac{f(2+\Delta x)-f(2)}{\Delta x}$ does NOT exist.

(b) $$f'(x) = \begin{cases} \dfrac{x^2-2x+2}{(x-1)^2} & \text{if } x > 2 \\ -\dfrac{x^2-2x+2}{(x-1)^2} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Solve $f'(x) > 0$ and $f'(x) < 0$

Ans : $f'(x) > 0$ when $x > 2$

$f'(x) < 0$ when $x < 2$ and $x \neq 1$

min $= (2, 0)$

(c) $f''(x) = \begin{cases} \dfrac{-2}{(x-1)^3} & \text{if } x > 2 \\ \dfrac{2}{(x-1)^3} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$

Solve $f''(x) > 0$ and $f''(x) < 0$

Ans : $f''(x) > 0$ when $1 < x < 2$

$f''(x) < 0$ when $x > 2$ or $x < 1$

point of inflection = $(2, 0)$

(d) vertical asymptote : $x = 1$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} -\dfrac{x(x-2)}{x-1} = -\infty$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} -\dfrac{x(x-2)}{x-1} = +\infty$

$f(x) = \begin{cases} \dfrac{x(x-2)}{x-1} & \text{if } x \geq 2 \\ -\dfrac{x(x-2)}{x-1} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$

oblique / horizontal asymptote :

① For $x \geq 2$, $f(x) = \dfrac{x(x-2)}{x-1}$

$m = \lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to +\infty} \dfrac{x-2}{x-1} = 1$

$c = \lim\limits_{x \to +\infty} f(x) - mx = \lim\limits_{x \to +\infty} \dfrac{x(x-2)}{x-1} - x = \lim\limits_{x \to +\infty} \dfrac{-x}{x-1} = -1$

$\therefore$ $y = x - 1$ is an oblique asymptote.

② For $x < 2$ and $x \neq 1$, $f(x) = -\dfrac{x(x-2)}{x-1}$

$m = \lim\limits_{x \to -\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to -\infty} -\dfrac{x-2}{x-1} = -1$

$c = \lim\limits_{x \to -\infty} f(x) - mx = \lim\limits_{x \to +\infty} -\dfrac{x(x-2)}{x-1} + x = \lim\limits_{x \to +\infty} \dfrac{x}{x-1} = 1$

$\therefore$ $y = -x + 1$ is an oblique asymptote.

Remark :

1) $m = \lim\limits_{x \to +\infty} \dfrac{f(x)}{x}$

$c = \lim\limits_{x \to +\infty} f(x) - mx$

If anyone of them does NOT exist, it means there is no oblique asymptote

2) If $m = 0$, the asymptote is horizontal

(e) x-intercept : Solve $f(x) = 0$

$\dfrac{x|x-2|}{x-1} = 0$

$x = 0$ or 2

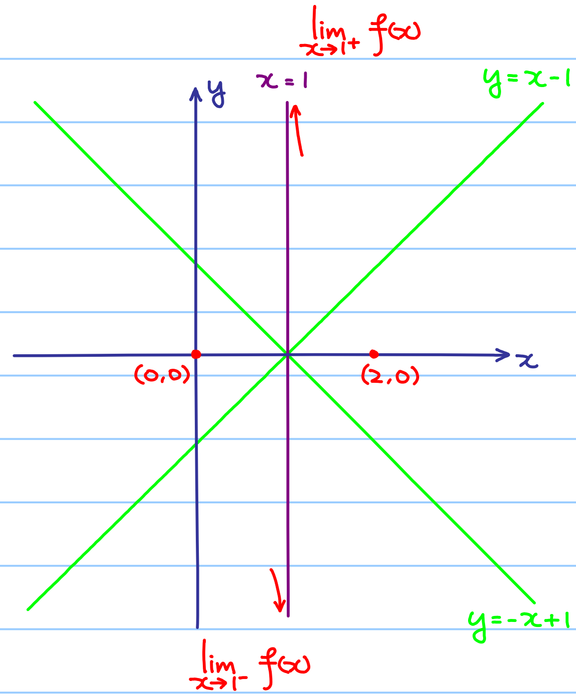
y-intercept : $f(0) = 0$.

(f) Sketch $y = f(x)$.

Step 1 : draw asymptotes

Step 2 : put down x-intercepts

and y-intercept

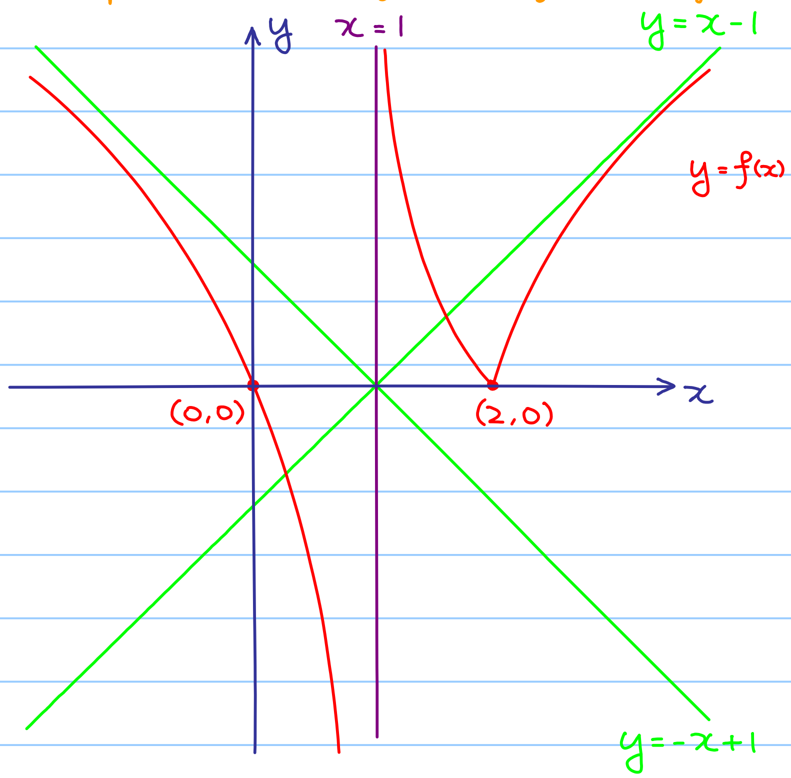


Step 3 :

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -\frac{x(x-2)}{x-1} = -\infty$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -\frac{x(x-2)}{x-1} = +\infty$$

Step 4 : Use the information $f'(x)$ and $f''(x)$



		1		2	
$f'(x)$	$-$	NOT defined	$-$	NOT defined	$+$
$\Downarrow$ $f(x)$	dec.		dec.		inc.

		1		2	
$f''(x)$	$-$	NOT defined	$+$	NOT defined	$-$
$\Downarrow$ $f(x)$	convex		concave		convex

Curve Sketching :

Goal : Given a function $f(x)$, sketch the graph of $y = f(x)$.

(Capturing main features)

- x-intercept — solve $f(x) = 0$
- y-intercept — y-intercept $= f(0)$
- increasing / decreasing — solve $f'(x) > 0$ / $f'(x) < 0$

  saddle point / max. / min. — change of sign of $f'(x)$?
- concave / convex — solve $f''(x) > 0$ / $f''(x) < 0$

  point of inflection — change of sign of $f''(x)$?
- vertical asymptote — any $x = a$ with $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$
- horizontal asymptote — $m = \lim_{x \to +\infty} \frac{f(x)}{x}$
- oblique asymptote — $c = \lim_{x \to +\infty} f(x) - mx$